Higgs bundles & crepant resolutions

$X / \mathbb{C}$ smth proj. curve / alg. closed field (of char. zero, suff. high)

<u>Defn:</u> (Hitchin, Simpson) pair (E, θ), E v bdle on X & sheaf of 1-forms

$\theta : E \to E \otimes \Omega^1_X$

Higgs field

<u>Hitchin map:</u> sends a Higgs bdle to the char. polynomial of "twisted" endomorphism θ

$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$, $a_i \in H^0(X, (\Omega^1_X)^{\otimes n-i})$

$A := \bigoplus_{i=0}^{n-1} H^0(X, (\Omega^1_X)^{\otimes n-i})$

Hitchin map: $\chi : \mathcal{M}_{Dol} \longrightarrow A$

stack of Higgs bdles

Examples / Facts:

- Rank 1 bdles:

$$\mathcal{M}_{Dol}^{GIT}(1,0) \overset{\text{rank degree}}{} \cong \underline{T^* J_X} = \underline{J_X \times H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)}$$

Jacobian

- $\chi : \mathcal{M}_{Dol}^{GIT}(n,d) \longrightarrow A$ proper & gen. fibre is an abelian variety

- (Non-abelian Hodge theory, Simpson et al)

$$\mathcal{M}_{Dol}^{GIT}(n,0) \overset{\text{real analyt.}}{\cong} \mathcal{M}_{dR}^{GIT}(n,0) \overset{\text{cplx analytic.}}{\cong} \pi_1(X)\text{-rep}(n)$$

$\mathcal{M}_{dR}$ $\shortparallel$ moduli space of flat alg. connections

- <u>2-dim'l examples:</u> X elliptic curve, $n=1$, $T^*X \to A$ trivial fibration

too easy

Conj: (Boalch) If M is a moduli space of meromorphic Higgs bdl(e)s, s.t. $\dim M = 2$, then $M^{[n]}$ is diffeomorphic to a moduli space of meromorphic Higgs bundles.

Analytical evidence + physics

Thm: (Groechenig–Mehrotra(?)–Rubtsov) If X is an elliptic curve, then $T^*X^{[n]}$ is equivalent to a moduli space of parabolic Higgs bdles.

at certain marked points, Higgs field may have order one singularities + flag at marked points

Thm (G.) Let X be an elliptic curve

(a) If Γ is a finite group acting on X, then the Γ-Hilbert scheme $(T^*X)^{[\Gamma]}$ is a moduli space of Higgs bundles on the orbifold $[X/\Gamma]$.

~~(b)~~ **Examples:**

- $\mathbb{Z}/2 \circlearrowright X$ $\quad (x \mapsto -x)$

- X cplx multiplication

  $\mathbb{Z}[i] \subset \mathbb{C}$ $\circlearrowleft$ $\mathbb{Z}/4$, $\quad \mathbb{Z}/3, \mathbb{Z}/6$ symmetries, hexagonal lattice

What is Γ-Hilbert scheme?

$\Gamma \circlearrowright Y$ surface, $\quad Y^{[\Gamma]} \subset Y^{|\Gamma|}$

$Y^{[\Gamma]}$: Γ-invariant $\mathcal{O}(\mathbb{C})$ (regular) $\quad$ $Y^{|\Gamma|}$: subschemes of length $|\Gamma|$

$Y^{[\Gamma]} \longrightarrow \underline{Y/\Gamma}$ GIT singular

"crepant resolution"

Thm (cont.)

(a) $T^*X^{[\Gamma]} \cong$ moduli space of Higgs bundles on orbicurve $[X/\Gamma]$

(b) $(T^*X)^{[\Gamma][n]} \cong$ moduli space of Higgs bdles on some weighted curve

(c) Hitchin map in all these cases factors through Hilbert–Chow morphism

- $Y^{[\Gamma]} \longrightarrow Y/\Gamma$
- $Y^{[n]} \longrightarrow Y^{(n)} = Y^n/S_n$.

$[X/\Gamma]$ orbicurve (DM-stack) $\rightsquigarrow$ weighted curve (X/Γ + weights, curve moduli space)

Motivation

(non-abelian Hodge theory)

First approximation to $Y^{[n]}$: stack of length n torsion sheaves on surface Y

$\mathrm{Tors}^n(Y)$

$$Y^{[n]} \longrightarrow \mathrm{Tors}^n(Y)$$

$$\begin{pmatrix} \mathcal{O}_Y \\ \downarrow \\ \mathcal{O}_T \end{pmatrix} \longmapsto \mathcal{O}_T$$

$$T^*X \overset{\text{diffeo.}}{\cong} (\mathbb{C}^\times)^2 \qquad (X \text{ ellipt. curve})$$

$$\mathrm{Tors}^n(\underbrace{(\mathbb{C}^\times)^2}_{\text{flat conn. of rank 1.}}) = \left[\{(A,B) \in GL_n(\mathbb{C}) \mid AB = BA\} \big/ {}^{\mathrm{conj}} GL_n\mathbb{C} \right]$$

$$\mathbb{I}\!\mathbb{R}$$

$$\mathbb{Z}^2 \text{ rep } (n) \overset{\text{analyt.}}{\cong} M_{dR}(X,n)$$

local systems / flat conn. of rank n.

Parabolic bundles / Orbibundles

orbicurve = DM-stack smooth, dim one, proj. coarse moduli space, generically a scheme

Higgs bdles / v.bdles make sense on orbicurves

$[X/\Gamma]$ orbicurve

$\updownarrow$ Nasatyr–Steer, Furuta–Steer

Defn (parabolic (Higgs) bundles)

- Let $\hat{X} = (X, D)$ be a weighted curve (D effective divisor on X, $D = \sum_{i=1}^{n} p_i x_i$), then a parab. bdle on $\hat{X}$ is a pair $\hat{E} = (E, E_{\bullet})$, E v.bdle on X, and $E_{\bullet}$ is flag data, so for every $x \in D$

$$0 = E_{x0} \subset \dots \subset E_{xi} \subset E_{xi+1} \subset \dots \subset E_x$$

$E_x \leftarrow$ fiber of E at x.

$\shortparallel$

E_{xp}

• chain of locally free sheaves, indexed by $\mathbb{Z}^{|D|}$

$\mathbb{Z}^{|D|}$ has $\mathbb{Z}$-basis e_i, $i = 1, \dots, k$

$$\subset \mathcal{E}_{0e_i} \subset \mathcal{E}_{1e_i} \subset \mathcal{E}_{2e_i} \subset \dots \subset \mathcal{E}_{p_i e_i} = \mathcal{E}(p_i).$$

$\mathcal{E}_{0e_i} = \mathcal{E}$

isomorphisms away from x_i

$\boxed{\text{periodicity condition}}$

Examples: $X = \mathbb{D} = \operatorname{Spec} \mathbb{C}[[t]]$ formal disc

μ_r action by rth roots of unity

$$[\mathbb{D}/\mu_r] \xrightarrow{\tau} \hat{\mathbb{D}} = \operatorname{Spec} \mathbb{C}[[t^r]]$$

$$\tilde{E} \longmapsto \tau_* \tilde{E}$$

is just μ_r-equiv. v.b. on $\mathbb{D}$

compensate loss of info: on $\mathbb{D}$ we have

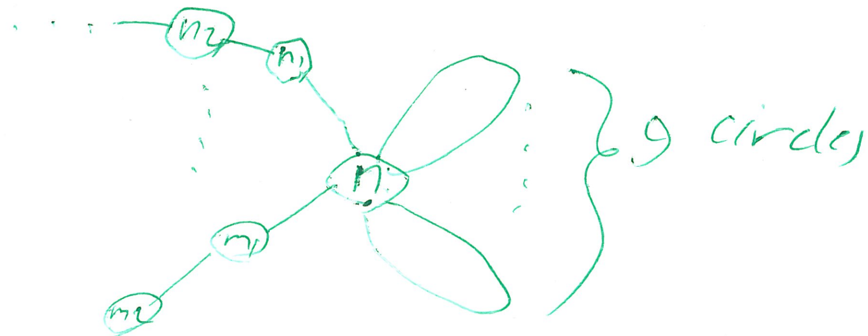
$\dots \subset \mathcal{O}(ix) \subset \mathcal{O}((i+1)x) \subset$

We obtain a chain of loc. free sheaves

$\tau_*(\tilde{E} \otimes \mathcal{O}(ix))$ on $\hat{\mathbb{D}}$.

Notation/numerical data:

- degree ✓
- rank of the bdle
- ranks describing the flags at every marked point



comet-shaped quiver

in our set-up:

X ellipt. curve

$\circlearrowleft$ $\mathbb{Z}/2$

$$[X/\mathbb{Z}/2] = \mathbb{P}^1 + \underbrace{4 \text{ marked points}}$$

$$\left(\begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[d, dash] & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 1 \\ & 1 & \end{tikzcd}\right)^{[n]} = \left(\begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[d, dash] & \\ & n \arrow[d, dash] & \\ n \arrow[r, dash] & 2n \arrow[r, dash] \arrow[d, dash] & n \\ & n & \end{tikzcd}\right)$$

Autoduality (of Hitchin systems)

classical limit of GL (could be stated for any pair of dual reductive groups)

Conj: (Donagi–Pantev)
There exists a (relative) FM kernel $\bar{P}$ on

$$\mathcal{M}_{Dol} \times_{A} \mathcal{M}_{Dol},$$

inducing an equivalence of derived categories

$$D^b(\mathcal{M}_{Dol}) \longrightarrow D^b(\mathcal{M}_{Dol})$$

+ satisfying the Hecke eigenproperty or (extending a generically defined kernel).

Evidence:

A ... space of char. polynomials

$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$ defines a curve $Y_a \subset T^*X$

"spectral curve".

A_{sm} ... open dense subset of smooth spectral curves

$\alpha \in A_{sm}$

$\mathcal{M}_{Dol,\alpha} \cong \mathrm{Pic}(Y_\alpha)$ = stack of line bdls on Y_α

By Mukai: $\mathcal{P}/\mathrm{Pic}(Y_\alpha) \times \mathrm{Pic}(Y_\alpha)$

induces an equiv.

$$D^b_{coh}(\mathrm{Pic}(Y_\alpha)) \longrightarrow D^b_{coh}(\mathrm{Pic}(Y_\alpha)).$$

$$\mathrm{Map}_{grp}(\mathrm{Pic}(Y_\alpha), B\mathbb{G}_m) \cong \mathrm{Pic}(Y_\alpha)$$

$\mathcal{P}/\mathcal{M}^{sm}_{Dol} \times_{A_{sm}} \mathcal{M}^{sm}_{Dol}$, extend to full base!

Done for integral spectral curves, by Arinkin (in char zero). (Hausel's $n!$-conj.)

Here: use autoduality / prove it for some cases
(weak)

- autoduality for $n=1$ (Mukai) to ~~Hilbert~~ Hilbert schemes & Higgs bdl(s)

- Bridgeland-King-Reid's McKay correspondence to verify autoduality for those nice examples.

X elliptic curve

Mukai: $D^b_{coh}(X) \cong D^b_{coh}(X)$ Segre ran the bdle
$\mathcal{O}_x \longmapsto \mathcal{L}$

base change $D^b_{coh}(T^*X) \cong D^b_{coh}(T^*X)$

1) At first we construct a map

$$\mathrm{Tors}^n(T^*X) \longrightarrow M_{Dol}(X)$$

2) ~~[illegible]~~ how do we have to change this to get a map from $(T^*X)^{[n]}$ to a mod. space of Higgs bdles?

1) **Defn:** Higgs bdls (E,θ) is admissible, if $\exists$ a filtration by Higgs subbdls, s.t. the successive quotients are line bdls of degree zero.

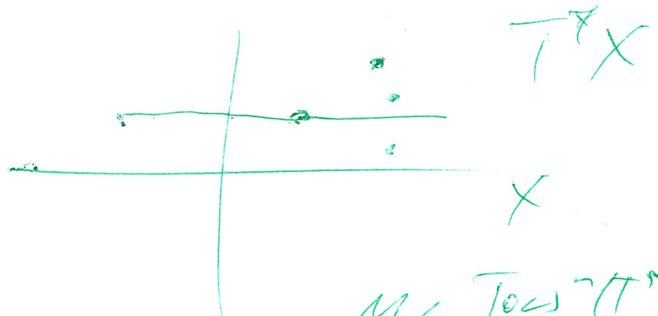
Prop: The equivalence (autoduality)

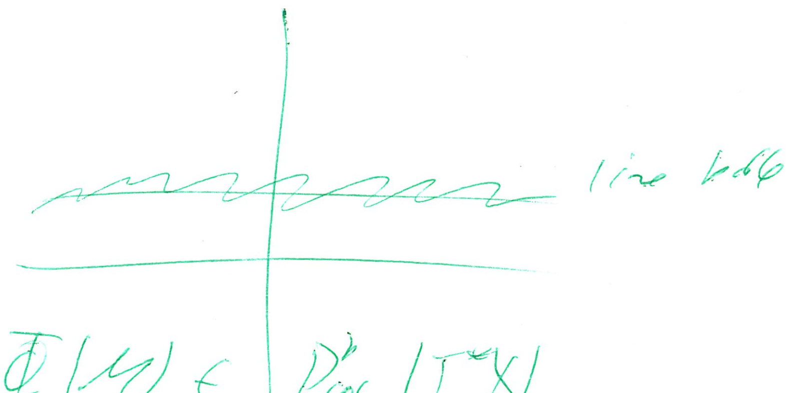
$$\Phi : D^b_{coh}(T^*X) \longrightarrow D^b_{coh}(T^*X)$$

induces an equiv of stacks

$$Tors^n(T^*X) \longrightarrow M^{ad}_{Dol}(X).$$

Proof:





$M \in Tors^n(T^*X)$, $\Phi(M) \in D^b_{coh}(T^*X)$.

2)
$$\begin{tikzcd}
(\mathcal{O}_{T^*X} \twoheadrightarrow M) \arrow[d, maps to] & (T^*X)^{[n]} \arrow[r] \arrow[d] & \mathcal{M}_{Dol}(X) \arrow[d] \\
M & \mathrm{Tors}^n(T^*X) \arrow[r, "\Phi"] & \mathcal{M}^{par}_{Dol}(X) \\
& & \Phi(M)
\end{tikzcd}$$

properness of Hitchin map is essential

$\Phi\left(\begin{matrix}\mathcal{O}_{T^*X} \\ \downarrow \\ M\end{matrix}\right)$ using Serre duality $\Downarrow$ endowing $\Phi(M)$ w/ a parabolic structure at $D \in X$.

Autoduality

- surfaces:

$$\begin{array}{ccccc}
D^b_{coh}(T^*X^{[\Gamma]}) & \overset{BKR}{=} & D([T^*X/\Gamma]) & = & D(T^*X)^{\Gamma} \\
\wr\wr & & & & \Vert \text{ Mukai's autoduality} \\
D(T^*X^{[\Gamma]}) & \overset{BKR}{=} & D([T^*X/\Gamma]) & = & D(T^*X)^{\Gamma}
\end{array}$$

- Hilbert scheme of surfaces $M, M^\vee$ $\leadsto$ FM-dual surfaces

$$\begin{array}{ccccc}
D(M^{[n]}) & \overset{BKR+Haiman}{=} & D([M^n/S_n]) & = & (D(M)^{\otimes n})^{S_n} \\
\wr\wr & & & & \Vert \\
D(M^{\vee[n]}) & = & D([M^{\vee n}/S_n]) & = & (D(M^\vee)^{\otimes n})^{S_n}.
\end{array}$$

Γ-action on an ∞-category $\mathcal{C}$

$B\Gamma$-indexed system of ∞-cat. $\mathcal{C}_?$

$$\circledcirc \longrightarrow B\Gamma$$

$$\mathcal{C}^{\Gamma} = \lim_{B\Gamma} \mathcal{C}_?$$